Heating rate of hadron beams during crystallization

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A theory concerning the relation between the heating rate and temperature of hadron beams is formulated from a quantum point of view. This theory predicts that the heating rate can be reduced by increasing the lattice periodicity of the accelerator with its fixed tunes and circumference. This prediction is quite consistent with simulation results.

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I. INTRODUCTION

Hadron beams are currently being accelerated by accelerators. The velocity of these beams is relatively small, which means that their Lorentz factor γ is not very large compared with that of electron beams. This is because the hadron has a large mass to be accelerated to the speed of light. In nonrelativistic hadron beams, the space-charge effect is serious. In space-charge effect-dominated beams, we expect that there occurs a special phenomenon, which is called a crystalline beam. Actually, it has been confirmed that this type of beam can be created based on molecular dynamics (MD) simulation [1–5].

The intrinsic temperature of a crystalline beam could be defined by the rms of the kinetic energy of the particles comprising the beam [4–6]. Thus, by reducing a temperature, we could obtain the space-charge-dominated beam, or a crystalline beam. It was also known that this type of crystalline beam can be destroyed by stopping the cooling system due to the alternating gradient focusing system [2,5]. In other words, a crystalline beam must be maintained by cooling the hadron beam. Further, it has been known that a crystalline beam cannot be obtained in a weak-focusing system [2,5]. Thus, we could not avoid this beam heating in accelerators. Temperature-heating rate relations could also be obtained in previous studies by simulations [5–9]. This relation has an interesting feature. At high temperatures, the heating rate increases as the temperature decreases. For lower temperatures, the heating rate decreases as the temperature decreases. The heating rate has a peak value for a certain temperature.

The above-mentioned feature of the heating rate can be understood qualitatively [9]. For a higher temperature, the intrabeam scattering effect is significant. Thus, the heating rate increases as the beam size becomes smaller by reducing the temperature. When the temperature is sufficiently lower, the intrabeam scattering effect is reduced. The heating rate cannot be higher. For a sufficiently low temperature, heating is caused by lattice vibration of the beam, because the hadron beam is crystallized at this low temperature.

For a higher temperature, there have been theories that deal with the intrabeam scattering effect. Pwinski first formulated such a theory, while considering the intrabeam scattering [10,11]. In this theory, the transverse emittances $\epsilon_{x,y}$ and the longitudinal emittance ϵ_s have a conservation law,

$$\epsilon_s\left(\frac{1}{\gamma^2}-\left\langle\frac{\eta^2}{\beta_x}\right\rangle\right)+\left\langle\frac{\epsilon_x}{\beta_x}\right\rangle+\left\langle\frac{\epsilon_y}{\beta_y}\right\rangle=\text{time independent}, \tag{1}$$

where $\beta_{x,y}$ are the transverse β functions, η is the dispersion function, and γ is the Lorentz factor. If this conservation law is exact, there is an equilibrium temperature. The time evolution of the temperature is confined by this law. Here, we can expect that the temperature will be in equilibrium exponentially. Thus, we may evaluate that the heating rate must be exactly zero for any temperature. However, the heating rate was positive definite according to previous MD simulations. Thus, we cannot apply this conservation law.

In order to create a crystalline beam, it is desirable to reduce the peak value of the heating rate that we mentioned previously. If this is possible, the necessary laser power will be lower. In order to solve this problem, it is necessary to establish a theory to explain the relation between the temperature and the heating rate. In Sec. II, we mention a theory to explain the simulation results. In Sec. II A, we evaluate the heating rate that is caused by intrabeam scattering. This is based on a theory established by Bjorken and Mtingwa [12]. In Sec. II B we evaluate the temperature where intrabeam scattering cannot occur. In Sec. II C, we formulate a theory of heating that is caused by the phonon emissions in a crystalline beam. Especially in Sec. II D, we explicitly calculate the heating rate by considering a one-dimension beam, because such a beam is so much simpler than the two or three-dimension case, and can thus deal with this case most rigorously. In Sec. III, we find that our theory is consistent with simulation results by comparing them. Conclusions and discussions are given in Sec. IV. In Appendix A, we show the diffusion time formula obtained by Bjorken and Mtingwa [12]. In Appendix B, we show the Taylor expansion of the Coulomb potential for a one-dimension crystal.

II. HEATING MECHANISM

A. Heating caused by intrabeam scattering

Using a formula of Bjorken and Mtingwa [12], we obtain an expression of the heating rate. For weak-focusing accelerators, we reproduce Eq. (1) from Eq. (A1) in Appendix A. According to Eq. (A6), $\lambda_1=\lambda_2=\lambda_3$ is the equilibrium point, which is equivalent to

$$\epsilon_s\left(\frac{1}{\gamma^2}-\frac{\eta^2}{\beta_x}\right)=\frac{\epsilon_x}{\beta_x}=\frac{\epsilon_y}{\beta_y}. \tag{2}$$

For strong-focusing accelerators, there is no conservation law, as they stated. Following the consideration for the weak-

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focusing case, we expect that the heating rate is determined by minimizing the value of Eq. (A6). This condition might mean that $\lambda_1=\lambda_2=\lambda_3$. However, this condition cannot be satisfied, because the β function $\beta_{x,y}(s)$ and the dispersion function $\eta(s)$ have an s dependence. Even if ϕ in Eq. (A1) is completely zero, the situation is the same. Since the heating is caused by the s dependence of the Twiss parameters, we expand Eq. (A6) around the mean of the Twiss parameters. In real accelerators, ϕ is sometimes very small. We also make an approximation for a small ϕ, obtaining the following formula:

$$\frac{d}{dt}\ln(\epsilon_s\epsilon_y\epsilon_s)=\frac{4}{15}\frac{\pi^2 z^2\alpha_e^2 MN(\log)}{\gamma\tilde{\Gamma}}\frac{1}{\overline{\beta}_x^2}\times\Bigg\{\Bigg\langle\Bigg[\frac{\gamma^2\overline{\eta}^2}{\overline{\beta}_x}\Bigg(2\frac{\delta\eta}{\overline{\eta}}-\frac{\delta\beta_x}{\overline{\beta}_x}\Bigg)-\delta\beta_x\Bigg]^2\Bigg\rangle+\Bigg\langle\Bigg(\delta\beta_x-\frac{\delta\beta_y}{\overline{\beta}_y}\overline{\beta}_x\Bigg)^2\Bigg\rangle+\Bigg\langle\Bigg[\frac{\overline{\beta}_x}{\overline{\beta}_y}\delta\beta_y-\frac{\gamma^2\overline{\eta}^2}{\overline{\beta}_x}\Bigg(2\frac{\delta\eta}{\overline{\eta}}-\frac{\delta\beta_x}{\overline{\beta}_x}\Bigg)\Bigg]^2\Bigg\rangle\Bigg\}, \tag{3}$$

where z is the charge of this particle evaluated by the electron charge, α_e is the fine-structure constant, M is the mass of the particle, N is the number of particles, (log) is the Coulomb log, $\tilde{\Gamma}=(2\pi)^{5/2}\epsilon_x\epsilon_y\sqrt{\epsilon_s}C$, C is the total length of this ring, $\overline{\beta}_x$, $\overline{\beta}_y$, and $\overline{\eta}$ are the mean values of each parameter, $\delta\beta_x=\beta_x-\overline{\beta}_x$, $\delta\beta_y=\beta_y-\overline{\beta}_y$, and $\delta\eta=\eta-\overline{\eta}$ are the deviations from each mean value.

By applying Eq. (3) to *a FODO lattice* (a lattice which is composed of focusing and defocusing magnets) using the thin-lens approximation, we obtain the following formula:

$$\frac{d}{dt}\ln(\epsilon_s\epsilon_y\epsilon_s)=\frac{4}{15}\frac{\pi^2\alpha_e^2 MN(\log)}{\gamma\tilde{\Gamma}}\frac{\mu^2}{2N_c^2}, \tag{4}$$

where N_c is the number of unit cells and μ is the tune. From Eq. (4), we find that we can reduce the heating rate by increasing N_c with fixed tunes.

B. Critical temperature

The heating curve has an interesting feature, as mentioned in the introduction. When the temperature is sufficiently high, the heating rate increases as the temperature decreases. When the temperature is sufficiently low, the heating rate decreases. If Eqs. (3) and (4) are correct for a lower temperature, the heating rate would be infinite for zero temperature. This means that the cause of the heating is not intrabeam scattering for a lower temperature. Actually, we expect that

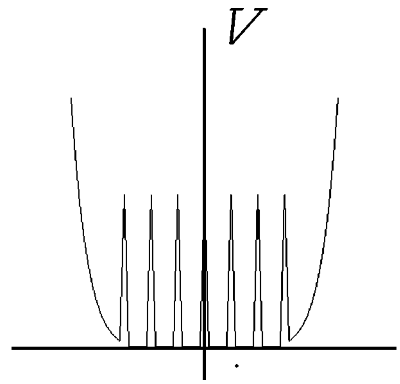

FIG. 1. Schematic picture of coulomb potential of a space-charge-dominant beam.

the space-charge effect is serious for a lower temperature. For a space-charge-dominant beam, intrabeam scattering cannot occur.

The temperature at which intrabeam scattering is not able to occur can be evaluated qualitatively. When the beam is space-charge dominant, the Coulomb force of this beam can almost be canceled by the external magnetic field, such that

$$\frac{z^2e^2N}{C}=\frac{M\omega_\beta^2R^2}{2}, \tag{5}$$

where e is the charge of an electron, ω_β is the betatron frequency, and R is the radius of this beam. By using Eq. (5), the mean distance d between particles is evaluated as follows:

$$d=\left(\frac{CR^2}{N}\right)^{1/3}=\left(\frac{z^2e^2}{M\omega_\beta^2}\right)^{1/3}. \tag{6}$$

Under this situation, we expect that the potential of the beam is described as in Fig. 1. In order for intrabeam scattering to occur, particles significantly approach each other. Thus, the kinetic energy must be larger than the potential energy. The critical temperature T_c beyond which there occurs intrabeam scattering is evaluated as

$$k_BT_c>\frac{z^2e^2}{d}=Mc^2\left(r_M\frac{\omega_\beta}{c}\right)^{2/3}, \tag{7}$$

where $r_M=z^2e^2/(Mc^2)$ and k_B is the Boltzmann constant.

C. Heating caused by phonon emissions

For a beam with a temperature lower than T_c, the cause of the heating is expected to be the lattice vibration of the crystallized beam [9]. Here, we consider the heating of this crystalline beam. This situation can also be described by the phonon emission from a quantum point of view. The time-dependent Hamiltonian can cause a phonon transition from one state to another. As time passes, if the number of phonons in higher states becomes larger than those in lower states, we consider that heating occurs. In this picture, the heating rate is described as follows:

$$\frac{1}{T}\frac{dT}{ds}=\sum_{n,m}\frac{v\hbar[(n_{+,1}\omega_{+,1}+\cdots+n_{3,N}\omega_{3,N})-(m_{+,1}\omega_{+,1}+\cdots+m_{3,N}\omega_{3,N})]}{k_B NT}$$
$$\times\lim_{t\to\infty}\frac{1}{t}\left|\left\langle n_{+,1},\ldots,n_{3,N}\right|P\exp\left[\frac{iMv}{\hbar}\int_0^t\mathcal{H}_I ds\right]\left|m_{+,1},\ldots,m_{3,N}\right\rangle\right|^2$$
$$\times\left(1-\exp\left[-\frac{v\hbar\omega_{+,1}}{k_BT}\right]\right)\cdots\left(1-\exp\left[-\frac{v\hbar\omega_{3,N}}{k_BT}\right]\right)\exp\left[-m_{+,1}\frac{v\hbar\omega_{+,1}}{k_BT}\cdots-m_{3,N}\frac{v\hbar\omega_{3,N}}{k_BT}\right], \quad (8)$$

where we assume the phonons obey the Boltzmann distribution, T is the temperature, $m_{i,j}=m_i(k_j)$ is the number of phonons in the initial states, $n_{i,j}=n_i(k_j)$ is the number of phonons in the final states, $\omega_{i,j}=\omega_i(k_j)$ is the lattice vibration "frequency," whose dimension is m^{-1}, i runs in $+$, $-$, and 3 and j runs from 1 to N, v is the velocity of the beam, P means the time-ordered product, $\mathcal{H}_{\mathcal{I}}$ is the interaction Hamiltonian that comes from the s dependence of the Hamiltonian, and $|n_{+,1},\ldots,n_{3,N}\rangle$'s and $|m_{+,1,},\ldots,m_{3,N}\rangle$'s are the phonon eigenstates.

In order to calculate the heating rate, we should calculate the phonon transition probability. Following, Wei, Li, and Sessler [1,5], the original Hamiltonian $\mathcal{H}$ can be written as

$$\mathcal{H}=\sum_{i=1}^{N}\left\{\frac{1}{2}(P_{ix}^2+P_{iy}^2+P_{iz}^2)-\frac{\gamma}{\rho(s)}x_iP_{iz}+\frac{1}{2}\left[\frac{[1-n(s)]}{\rho^2(s)}x_i^2+\frac{n(s)}{\rho^2(s)}y_i^2\right]\right\}+\frac{r_M}{\beta^2\gamma^2}V_c, \quad (9)$$

where

$$V_c(x_i,y_i,z_i)=\frac{1}{2}\sum_{i=1}^{N}\sum_{m\neq 0}\frac{1}{\sqrt{(x_{i+m}-x_i)^2+(y_{i+m}-y_i)^2+(z_{i+m}-z_i)^2}}, \quad (10)$$

i denotes the particle index, s is the path length of an ideal particle, $\rho(s)$ is the radius of curvature, $\beta=v/c$, x,y,z are the spatial coordinates, $n(s)$ is the strength of the focusing magnetic field, which is represented as $-\rho/(B_0)dB_x/dy=-\rho/(B_0)dB_y/dx$ and B_0 is the constant bending magnetic field in the y direction.

We expect that the heating rate must be reduced by increasing the number of cells N_c with a fixed tune, referring to intrabeam scattering theory [See Eq. (4)]. Following this speculation, we must select an interaction Hamiltonian $\mathcal{H}_I$ that approaches zero for the above infinite N_c. Thus, we must apply a canonical transformation to Eq. (9). First, we adopt the coordinates $\delta x_i,\delta y_i,\delta z_i$ and their canonical momentum $\delta p_{ix},\delta p_{iy},\delta p_{iz}$ that represent the deviations from the equilibrium positions x_i^0,y_i^0,z_i^0 and their momentum p_{ix}^0, p_{iy}^0, p_{iz}^0. For this purpose, we consider a canonical transformation that is obtained by the generating function,

$$W(\delta p_{ix},x_i,\delta p_{iy},y_i,\delta p_{iz},z_i,s)=(p_{ix}^0+\delta p_{ix})x_i-\delta p_{ix}x_i^0+(p_{iy}^0+\delta p_{iy})y_i-\delta p_{iy}y_i^0+(p_{iz}^0+\delta p_{iz})z_i-\delta p_{iz}z_i^0. \quad (11)$$

Here, $x_i^0,p_{ix}^0,y_i^0,p_{iy}^0,z_i^0,p_{iz}^0$, must satisfy the following equations of motion:

$$P_{ix}^{0\prime}=\frac{\gamma}{\rho(s)}P_{iz}^0-\frac{1-n(s)}{\rho^2(s)}x_i^0-\frac{r_M}{\beta^2\gamma^2}\frac{\partial}{\partial x_i^0}V_c,$$
$$x_i^{0\prime}=P_{ix}^0,$$
$$P_{iy}^{0\prime}=-\frac{n(s)}{\rho^2(s)}y_i^0-\frac{r_M}{\beta^2\gamma^2}\frac{\partial}{\partial y_i^0}V_c,$$
$$y_{iy}^{0\prime}=P_{iy}^0,$$
$$P_{iz}^{0\prime}=-\frac{r_M}{\beta^2\gamma^2}\frac{\partial}{\partial z_i^0}V_c,$$
$$z_i^{0\prime}=P_{iz}^0-\frac{\gamma}{\rho(s)}x_i^0, \quad (12)$$

where the prime denotes differentiation with s. From now on, we abbreviate the summation mark, Σ_i. Thus, we obtain the following Hamiltonian:

$$\mathcal{H}=\frac{1}{2}(\delta p_{ix}^2+\delta p_{iy}^2+\delta p_{iz}^2)-\frac{\gamma}{\rho(s)}\delta x_i\delta p_{iz}+\frac{1}{2}\left[\frac{[1-n(s)]}{\rho^2(s)}\delta x_i^2+\frac{n(s)}{\rho^2(s)}\delta y_i^2\right]+\frac{r_M}{\beta^2\gamma^2}\frac{1}{2}\delta\mathbf{x}V_0^{(2)}\delta\mathbf{x}(s)+\cdots, \quad (13)$$

where $V_0^{(2)}(s)$ is the second coefficient of the Taylor series and $\delta\mathbf{x}=(\delta x_1,\delta y_1,\delta z_1,\ldots,\delta x_N,\delta y_N,\delta z_N)$. Here, the coefficients of the Taylor series generally depend on s, because we expand V_c around the solutions of Eq. (12). The first coefficient of the Taylor series has been removed by Eq. (12).

In a strong-focusing system, $n(s)$ is discontinuous because the focusing system and defocusing system appear step by step. In order to obtain an interaction Hamiltonian that becomes smaller as N_c becomes larger, we need to make a canonical transformation, and rewrite the focusing strength using the β function, because the β function is continuous and it approaches its mean value for an infinite N_c. For this purpose, we consider a canonical transformation defined by the following generating function,

$$F[\delta x_i,\delta y_i,\psi_{ix},\psi_{iy}]=-\frac{\delta x_i^2}{2\beta_x(s)}[\tan\psi_{ix}+\alpha_x(s)]-\frac{\delta y_i^2}{2\beta_y(s)}[\tan\psi_{iy}+\alpha_y(s)]. \tag{14}$$

The Hamiltonian is rewritten as

$$\mathcal{H}=\frac{J_{ix}}{\beta_x(s)}+\frac{J_{iy}}{\beta_y(s)}+\frac{1}{2}\delta p_{iz}^2-\frac{\gamma}{\rho(s)}\delta x_i\delta p_{iz}+\frac{r_M}{\beta^2\gamma^2}\frac{1}{2}\delta\mathbf{x}V_0^{(2)}(s)\delta\mathbf{x}+\cdots, \tag{15}$$

where

$$\delta x_i=\sqrt{2\beta_x(s)J_{ix}}\cos\psi_{ix},$$
$$\delta p_x=-\left(\frac{2J_{ix}}{\beta_x(s)}\right)^{1/2}[\sin\psi_{ix}+\alpha_x(s)\cos\psi_{ix}],$$
$$\delta y_i=\sqrt{2\beta_y(s)J_{iy}}\cos\psi_{iy},$$
$$\delta p_y=-\left(\frac{2J_{iy}}{\beta_y(s)}\right)^{1/2}[\sin\psi_{iy}+\alpha_y(s)\cos\psi_{iy}]. \tag{16}$$

The Twiss parameters satisfy the following equations:

$$\frac{d^2}{ds^2}\sqrt{\beta_x(s)}+\frac{[1-n(s)]}{\rho^2(s)}\sqrt{\beta_x(s)}-\frac{1}{[\sqrt{\beta_x(s)}]^3}=0,$$
$$\frac{d^2}{ds^2}\sqrt{\beta_y(s)}+\frac{n(s)}{\rho^2(s)}\sqrt{\beta_y(s)}-\frac{1}{[\sqrt{\beta_y(s)}]^3}=0, \tag{17}$$

where $\alpha_{x,y}=-\beta'_{x,y}/2$. We change the above action-angle variables to the coordinates and their canonical momentum. We now consider the following generating function:

$$F_2[X_i,Y_i,\psi_{ix},\psi_{iy}]=-\frac{X_i^2}{2}\tan\psi_{ix}-\frac{Y_i^2}{2}\tan\psi_{iy}. \tag{18}$$

The Hamiltonian is rewritten as

$$\mathcal{H}=\frac{1}{2\beta_x(s)}(X_i^2+P_{iX}^2)+\frac{1}{2\beta_y(s)}(Y_i^2+P_{iY}^2)+\frac{\delta p_{ix}^2}{2}-\frac{\gamma}{\rho(s)}\sqrt{\beta_x(s)}X_i\delta p_{iz}+\frac{r_M}{\beta^2\gamma^2}\frac{1}{2}\delta\mathbf{x}V_0^{(2)}(s)\delta\mathbf{x}+\cdots, \tag{19}$$

where $\delta\mathbf{x}=(\sqrt{\beta_x(s)}X_1, \sqrt{\beta_y(s)}Y_1,\delta z_1,\ldots,\sqrt{\beta_x(s)}X_N, \sqrt{\beta_y(s)}Y_N,\delta z_N)$. We have now succeeded to rewrite the Hamiltonian in terms of β functions.

D. Explicit calculation for the one-dimension case

In order to proceed further, we must calculate the coefficients of the Taylor series explicitly. Here, we discuss how to calculate the heating rate. For simplicity, we consider the case that the crystal beam has a one-dimensional structure. Actually, when the number of particles N is smaller than the critical value, the crystallized beam has a structure such chain. In Appendix B, we present a Taylor series of the Coulomb potential where the hadron beam has a one-dimensional structure.

In order to define the phonon frequency, we initially consider the Hamiltonian up to second order. Afterwards, it will be clear that we must consider the fourth order of magnitude. According to Eq. (B1), Eq. (19) is rewritten as

$$\mathcal{H}_{\mathrm{sec}}=\int dk\Bigg\{\frac{1}{\beta_x(s)}p_k^\dagger p_k+\left(\frac{1}{\beta_x(s)}-\beta_x(s)\Omega_k^2\right)\xi_k^\dagger\xi_k+\frac{1}{\beta_y(s)}u_k^\dagger u_k+\left(\frac{1}{\beta_y(s)}-\beta_y(s)\Omega_k^2\right)\eta_k^\dagger\eta_k-\frac{\gamma}{\rho(s)}\sqrt{\beta_x(s)}(\xi_k^\dagger q_k+q_k^\dagger\xi_k)+q_k^\dagger q_k+2\Omega_k^2\zeta_k^\dagger\zeta_k\Bigg\}, \tag{20}$$

where

$$\Omega_k^2=\frac{r_M}{\beta^2\gamma^2}2\sum_{n=1}^{\infty}\frac{1-\cos(kn\Delta)}{n^3\Delta^3}, \tag{21}$$

Δ is the distance between the nearest-neighbor particles and k moves from $-\pi/\Delta$ to π/Δ, we then make a summation for an infinite number of particles and use the following Fourier integrals:

$$X_m=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(\xi_k\exp[ikm\Delta]+\xi_k^\dagger\exp[-ikm\Delta]),$$
$$P_{mX}=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(p_k\exp[ikm\Delta]+p_k^\dagger\exp[-ikm\Delta]),$$
$$Y_m=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(\eta_k\exp[ikm\Delta]+\eta_k^\dagger\exp[-ikm\Delta]),$$
$$P_{mY}=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(u_k\exp[ikm\Delta]+u_k^\dagger\exp[-ikm\Delta]),$$
$$\delta z_m=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(\zeta_k\exp[ikm\Delta]+\zeta_k^\dagger\exp[-ikm\Delta]),$$
$$\delta p_{mz}=\int dk\left(\frac{\Delta}{2(2\pi)}\right)^{1/2}(q_k\exp[ikm\Delta]+q_k^\dagger\exp[-ikm\Delta]),$$
$$\delta_p(\theta)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in\theta}, \tag{22}$$

where $\delta_p(\theta)$ is the periodic δ function.

We divide Eq. (20) into a harmonic part $\mathcal{H}_0$ that is independent of s and an inharmonic part $\mathcal{H}_I$

$$\mathcal{H}_0=\int dk\left\{\frac{\mu_x}{C}p_k^\dagger p_k+\left(\frac{\mu_x}{C}-\overline{\beta}_x\Omega_k^2\right)\xi_k^\dagger\xi_k+\frac{\mu_y}{C}u_k^\dagger u_k+\left(\frac{\mu_y}{C}-\overline{\beta}_y\Omega_k^2\right)\eta_k^\dagger\eta_k-\gamma\left\langle\frac{\sqrt{\beta_x(s)}}{\rho(s)}\right\rangle(\xi_k^\dagger q_k+q_k^\dagger\xi_k)+q_k^\dagger q_k+2\Omega_k^2\zeta_k^\dagger\zeta_k\right\}, \tag{23}$$

$$\mathcal{H}_I=\int dk\left\{\left(\frac{1}{\beta_x(s)}-\frac{\mu_x}{C}\right)p_k^\dagger p_k+\left(\frac{1}{\beta_x(s)}-\beta_x(s)\Omega_k^2-\frac{\mu_x}{C}+\overline{\beta}_x\Omega_k^2\right)\xi_k^\dagger\xi_k+\left(\frac{1}{\beta_y(s)}-\frac{\mu_y}{C}\right)u_k^\dagger u_k+\left(\frac{1}{\beta_y(s)}-\beta_y(s)\Omega_k^2-\frac{\mu_y}{C}+\overline{\beta}_y\Omega_k^2\right)\eta_k^\dagger\eta_k+\left(-\gamma\frac{\sqrt{\beta_x(s)}}{\rho(s)}+\gamma\left\langle\frac{\sqrt{\beta_x(s)}}{\rho(s)}\right\rangle\right)(\xi_k^\dagger q_k-q_k^\dagger\xi_k)\right\}. \tag{24}$$

The mixing term in $\mathcal{H}_0$ can be eliminated by canonical transformations. The canonical transformation generated by a generating function,

$$W[\xi_k,\eta_k,\zeta_k,\bar{p}_k^\dagger,\bar{u}_k^\dagger,\bar{q}_k^\dagger]=-\sqrt{2\Omega_k^2}\zeta_k\bar{q}_k^\dagger+\sqrt{\frac{C}{\mu_x}}\xi_k\bar{p}_k^\dagger+\sqrt{\frac{C}{\mu_y}}\eta_k\bar{u}_k^\dagger \tag{25}$$

rewrites the Hamiltonian $\mathcal{H}_0$ as

$$\mathcal{H}_0=\int dk\left[\bar{\bar{p}}_k^\dagger\bar{\bar{p}}_k+\left(\frac{\mu_x}{C}-\overline{\beta}_x\Omega_k^2\right)\frac{\mu_x}{C}\bar{\bar{\xi}}_k^\dagger\bar{\bar{\xi}}_k+2\Omega_k^2\bar{\bar{q}}_k^\dagger\bar{\bar{q}}_k+\bar{\bar{\zeta}}_k^\dagger\bar{\bar{\zeta}}_k+\gamma\left\langle\frac{\sqrt{\beta_x(s)}}{\rho(s)}\right\rangle\sqrt{2\Omega_k^2}\left(\frac{\mu_x}{C}\right)^{1/2}(\bar{\bar{q}}_k^\dagger\bar{\bar{\xi}}_k+\bar{\bar{\xi}}_k^\dagger\bar{\bar{q}}_k)+\bar{\bar{u}}_k^\dagger\bar{\bar{u}}_k+\left(\frac{\mu_y}{C}-\overline{\beta}_y\Omega_k^2\right)\frac{\mu_y}{C}\bar{\bar{\eta}}_k^\dagger\bar{\bar{\eta}}_k\right]. \tag{26}$$

We exchange the role of $\bar{\bar{\zeta}}_k$ and its canonical momentum $\bar{\bar{q}}_k^\dagger$ by using a canonical transformation whose generating function is

$$W=\bar{\bar{\zeta}}_k\overline{\zeta_k}. \tag{27}$$

We obtain the following $\mathcal{H}_0$:

$$\mathcal{H}_0=\int dk\left[\bar{\bar{p}}_k^\dagger\bar{\bar{p}}_k+\left(\frac{\mu_x}{C}-\overline{\beta}_x\Omega_k^2\right)\frac{\mu_x}{C}\bar{\bar{\xi}}_k^\dagger\bar{\bar{\xi}}_k+\overline{q_k^\dagger}\,\overline{q_k}+2\Omega_k^2\overline{\zeta_k^\dagger}\,\overline{\zeta_k}+\gamma\left\langle\frac{\sqrt{\beta_x(s)}}{\rho(s)}\right\rangle\sqrt{2\Omega_k^2}\sqrt{\frac{\mu_x}{C}}(\overline{\zeta_k}\bar{\bar{\xi}}_k+\bar{\bar{\xi}}_k^\dagger\overline{\zeta_k^\dagger})+\bar{\bar{u}}_k^\dagger\bar{\bar{u}}_k+\left(\frac{\mu_y}{C}-\overline{\beta}_y\Omega_k^2\right)\frac{\mu_y}{C}\bar{\bar{\eta}}_k^\dagger\bar{\bar{\eta}}_k\right]. \tag{28}$$

In this $\mathcal{H}_0$, the mixing term between the coordinates and its momentum disappears. The mixing term only appears in a potential term. This kind of mixing term can be eliminated by a rotation transformation, which can be obtained by the generating function,

$$W[\bar{\bar{p}}_k^\dagger,\overline{q_k},\tilde{\xi}_k,\tilde{\zeta}_k^\dagger]=-(\cos\theta_0^k\tilde{\xi}_k-\sin\theta_0^k\tilde{\zeta}_k^\dagger)\bar{\bar{p}}_k^\dagger-(\sin\theta_0^k\tilde{\xi}_k)+\cos\theta_0^k\tilde{\zeta}_k^\dagger)\overline{q_k}, \tag{29}$$

where θ_0^k satisfies

$$\tan 2\theta_0^k=\frac{2\gamma\sqrt{2\Omega_k^2}}{-2\Omega_k^2+\left(\frac{\mu_x}{C}-\overline{\beta}_x\Omega_k^2\right)\frac{\mu_x}{C}}\sqrt{\frac{\mu_x}{C}}\left\langle\frac{\sqrt{\beta_x(s)}}{\rho(s)}\right\rangle, \tag{30}$$

where θ_0^k is selected in order to eliminate the mixing term.

Since $\mathcal{H}_0$ is diagonalized, we can quantize this system as follows:

$$\tilde{\xi}_k=\left(\frac{\hbar}{2Mv\,\omega_k^{(+)}}\right)^{1/2}e^{-i\omega_k^{(+)}s}a_k^{(+)},$$

$$\tilde{p}_k^\dagger=i\left(\frac{\hbar\,\omega_k^{(+)}}{2Mv}\right)^{1/2}e^{i\omega_k^{(+)}s}a_k^{(+)\dagger},$$

$$\bar{\bar{\eta}}_k=\left(\frac{\hbar}{2Mv\,\omega_k^{(3)}}\right)^{1/2}e^{-i\omega_k^{(3)}s}a_k^{(3)},$$

$$\bar{\bar{u}}_k^\dagger=i\left(\frac{\hbar\,\omega_k^{(3)}}{2Mv}\right)^{1/2}e^{i\omega_k^{(3)}s}a_k^{(3)\dagger},$$

$$\tilde{\xi}_k=\left(\frac{\hbar}{2Mv\,\omega_k^{(-)}}\right)^{1/2}e^{-i\omega_k^{(-)}s}a_k^{(-)},$$

$$\tilde{q}_k^\dagger=i\left(\frac{\hbar\,\omega_k^{(-)}}{2Mv}\right)^{1/2}e^{i\omega_k^{(-)}s}a_k^{(-)\dagger}, \tag{31}$$

where

$$\omega_k^{(\pm)2}=\frac{1}{2C^2}\left\{\mu_x^2+2C^2\Omega_k^2-C\overline{\beta}_x\mu_x\Omega_k^2\pm\left\{(\mu_x^2+2C^2\Omega_k^2-C\overline{\beta}_x\mu_x\Omega_k^2)^2-8C^2\Omega_k^2\mu_x\times\left(-\left\langle\frac{\sqrt{\beta_x}}{\rho(s)}\right\rangle^2C\gamma^2+\mu_x-C\overline{\beta}_x\Omega_k^2\right)\right\}^{1/2}\right\},$$

$$\omega_k^{(3)2} = \left(\frac{\mu_y}{C} - \overline{\beta}_y \Omega_k^2\right)\frac{\mu_y}{C},$$

$$[a_k^{(+)}, a_{k'}^{(+)\dagger}] = \delta(k-k'),$$

$$[a_k^{(3)}, a_{k'}^{(3)\dagger}] = \delta(k-k'),$$

$$[a_k^{(-)}, a_{k'}^{(-)\dagger}] = \delta(k-k'). \tag{32}$$

In order that the crystalline beam is always stable, the β functions must satisfy

$$\mu_x > \left\langle \frac{\sqrt{\beta_x(s)}}{\rho(s)} \right\rangle^2 C\gamma^2 + C\overline{\beta}_x \Omega_k^2, \quad \frac{1}{\Omega_k} > \overline{\beta}_y. \tag{33}$$

This generalizes the condition that the ring must be operated below the transition energy [2]. For this diagonalized $\mathcal{H}_0$, Eq. (24) is rewritten using the following relation:

$$p_k^\dagger = \left(\frac{C}{\mu_x}\right)^{1/2} [\cos\theta_0^k \tilde{p}_k^\dagger - \sin\theta_0^k \tilde{q}_k],$$

$$q_k^\dagger = -\sqrt{2\Omega_k^2}[\sin\theta_0^k \tilde{\xi}_k^\dagger + \cos\theta_0^k \tilde{\zeta}_k],$$

$$u_k^\dagger = \left(\frac{C}{\mu_y}\right)^{1/2} \bar{\bar{u}}_k^\dagger,$$

$$\xi_k = \left(\frac{\mu_x}{C}\right)^{1/2} [\cos\theta_0^k \tilde{\xi}_k - \sin\theta_0^k \tilde{\zeta}_k^\dagger],$$

$$\zeta_k = \frac{1}{\sqrt{2\Omega_k^2}} [\sin\theta_0^k \tilde{p}_k + \cos\theta_0^k \tilde{q}_k^\dagger],$$

$$\eta_k = \left(\frac{\mu_y}{C}\right)^{1/2} \bar{\bar{\eta}}_k. \tag{34}$$

Since we have obtained $\mathcal{H}_I$, we can calculate the transition probability. According to Eqs. (24) and (34), $\int \mathcal{H}_I ds$ is written as

$$\int_0^\infty \mathcal{H}_I ds = \sum_k \int_0^\infty [f(s) - \overline{f}] \exp(i\omega_k s) ds$$
$$= \sum_k \int_0^\infty \sum_{m\neq 0} f_m \exp\left(i\frac{2\pi m}{C/N_c} s\right) \exp(i\omega_k s) ds, \tag{35}$$

where $f(s)$ is a generic periodic function of the lattice length, $L = C/N_c$, f_m is the Fourier coefficient, and ω_k stands for the sum of the two-phonon frequency, which are chosen among $\omega_k^{(\pm)}$ and $\omega_k^{(3)}$. Performing the s integration in Eq. (35) yields the δ function. Thus, we obtain the following resonance condition:

$$N_c \frac{2\pi m}{C} + \omega_k = 0 \quad (\text{arbitrary integer except } m=0). \tag{36}$$

Here, we consider whether heating occurs or not. We can roughly estimate ω_k as

$$\omega_k \sim 2\frac{\mu_x}{C}. \tag{37}$$

According to Eq. (35), for the lowest order case

$$\frac{\mu_x}{C} \sim \pi. \tag{38}$$

In the FODO cell, the system is unstable near $\mu_x/C \sim \pi$. Thus, we cannot satisfy this condition. Further, the phase advance is usually chosen to be

$$\frac{\mu_x}{C} \leqslant \frac{\pi}{2}, \tag{39}$$

because the system can be unstable due to the space-charge effect. This means that the heating cannot occur for the one-dimension case up to the second order of the Hamiltonian.

In order to know the T dependence of the heating rate, we must calculate the higher order of the transition amplitude. It is necessary to know the relation between the order of T in the heating rate and the order of the transition amplitude. For this purpose, we see the $\hbar$ dependence in the heating rate. We consider the heating that comes from the n transition of phonons. When we consider the first order of ds in the transition matrix as this n-phonon transition heating, the transition matrix is written as

$$\sim \langle m \pm n | \int \frac{ds}{\hbar} \hbar^{n/2} (a^n + a^{\dagger n}) | m \rangle. \tag{40}$$

Following Eq. (8), the heating rate is written as

TABLE I. Parameters for these accelerators and properties of the hadron beam.

Circumference C	8π m
Local radius of curvature ρ	1 m
Horizontal tune $\mu_x = 2\pi\nu_x$	12.9
Vertical tune $\mu_y = 2\pi\nu_y$	12.4
Number density of this beam	31 860.9 m^{-1}
Lorentz factor γ	1.000 0442
Ion species	$^{24}Mg^+$
Superperiod N_c	9
Unit length l_f	$2\pi/45$ m
Focusing force K_F	1.227 m^{-2}
Defocusing force K_D	−2.0505 m^{-2}
Superperiod N_c	10
Unit length l_f	$\pi/25$ m
Focusing force K_F	1.5 m^{-2}
Defocusing force K_D	−2.3 m^{-2}
Superperiod N_c	20
Unit length l_f	$\pi/50$ m
Focusing force K_F	3.77 m^{-2}
Defocusing force K_D	−4.55 m^{-2}

$$\frac{dT}{ds}\sim\hbar^{n-2}\frac{\exp\left(n\frac{\hbar v\omega}{k_BT}\right)-1}{\left(-1+\exp\left[\frac{\hbar v\omega}{k_BT}\right]\right)^n}n\hbar v\omega$$

$$\to\frac{(n\omega)^2}{\omega^n}T^{n-1}\quad(\text{for }\ \hbar\to 0).\tag{41}$$

Thus, dT/ds is proportional to T for $n=2$. For a one-dimension beam this coefficient is zero. In order to calculate the T dependence in the heating rate, we must consider the higher phonon process. We should notice that the heating effect becomes smaller as the order of the phonon transition becomes larger in the lower temperature region.

Here, we consider the effect of the higher order of the Coulomb potential. We expect that a similar condition to Eq. (36) will be obtained. For example, we consider the case that only four $\omega_k^{(+)}$'s satisfy the resonance condition: $\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_3}^{(+)}+\omega_{k_4}^{(+)}=N_c 2\pi/C$. Actually, we can consider the accelerator that this resonance condition satisfies ($N_c=9$ in Table I). When we calculate the heating rate, we better consider the dispersion relation that frequencies must satisfy, which is determined by each accelerator. The term that in the Coulomb potential is important is determined by each accelerator.

According to Eq. (B1) in Appendix B, the fourth-order interaction term $V_c^{(4)}$ that is concerned with four $\omega_k^{(+)}$'s, are written as

$$V_c^{(4)}=\sum_{m\neq 0,n}\frac{r_M}{\beta^2\gamma^2}\frac{1}{48}\left(\frac{9\beta_x^2(s)(X_{n+m}-X_n)^4}{|m|^5\Delta^5}-\frac{72\beta_x(s)(X_{n+m}-X_n)^2(\delta z_{n+m}-\delta z_n)^2}{|m|^5\Delta^5}+\frac{24(\delta z_{n+m}+\delta z_n)^2(\delta z_{n+m}-\delta z_n)^2}{|m|^5\Delta^5}\right),\tag{42}$$

where the third term has nothing to do with the heating, because there is no β function in this term. We write the part of $\int ds/(\hbar/Mv)iV_c^{(4)}$, which is proportional to $a_{k_1}^{(+)}a_{k_2}^{(+)}a_{k_3}^{(+)}a_{k_4}^{(+)}$ and $a_{k_1}^{\dagger(+)}a_{k_2}^{(+)}a_{k_3}^{(+)}a_{k_4}^{(+)}$

$$\int_{-\infty}^{\infty}\frac{ds}{\hbar/(Mv)}V_c^{(4)}=\sum_{m\neq 0,m,l=-\infty}^{\infty}i\frac{r_M}{192(2\pi)^2\beta^2\gamma^2\Delta^3}\int dk_1\,dk_2\,dk_3\,dk_4\times\frac{2\pi}{\Delta}\delta_p(k_1+k_2+k_3+k_4)\times(2\pi)\delta\left(\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_3}^{(+)}+\omega_{k_4}^{(+)}+\frac{2\pi l}{C/N_c}\right)\frac{(e^{ik_1m\Delta}-1)(e^{ik_2m\Delta}-1)(e^{ik_3m\Delta}-1)(e^{ik_4m\Delta}-1)}{|m|^5}\times\left(\frac{\hbar}{Mv}(9A_l+72B_l)a_{k_1}^{(+)}a_{k_2}^{(+)}a_{k_3}^{(+)}a_{k_4}^{(+)}+\text{H.c.}\right),\tag{43}$$

where

$$A_l=\frac{1}{4}\left(\frac{\mu_x}{C}\right)^2\frac{\cos\theta_0^{k_1}\cos\theta_0^{k_2}\cos\theta_0^{k_3}\cos\theta_0^{k_4}}{\sqrt{\omega_{k_1}^{(+)}\omega_{k_2}^{(+)}\omega_{k_3}^{(+)}\omega_{k_4}^{(+)}}}b_l^{(2)},$$

$$B_l=\frac{1}{8}\frac{\mu_x}{C}\cos\theta_0^{k_1}\cos\theta_0^{k_2}\sin\theta_0^{k_3}\sin\theta_0^{k_4}\times\frac{\sqrt{\omega_{k_3}^{(+)}\omega_{k_4}^{(+)}}}{\sqrt{\omega_{k_1}^{(+)}\omega_{k_2}^{(+)}\Omega_{k_3}^2\Omega_{k_4}^2}}b_l^{(1)},\tag{44}$$

$\theta_0^{k_1},\theta_0^{k_2},\theta_0^{k_3},\theta_0^{k_4}$ can be calculated by Eq. (30), $b_l^{(2)}$ and $b_l^{(1)}$ are Fourier coefficients that are defined by

$$\beta_x(s)=\sum_l\ \exp\left(i\frac{2\pi l}{C/N_e}s\right)b_l^{(1)},$$

$$\beta_x^2(s)=\sum_l\ \exp\left(i\frac{2\pi l}{C/N_c}\right)b_l^{(2)},\tag{45}$$

l is an integer. In order to calculate the matrix elements, we replace the phonon creation and annihilation operators as follows:

$$a_{k_1}^{(+)}\to\sqrt{\frac{C}{2\pi}}\alpha_{\kappa_1}^{(+)},\quad a_{k_1}^{\dagger(+)}\to\sqrt{\frac{C}{2\pi}}\alpha_{\kappa_1}^{\dagger(+)},\tag{46}$$

where κ_1 is a discrete variable that moves from $-\pi/\Delta$ to π/Δ in $2\pi/(N\Delta)$ steps and the operators satisfy

$$\alpha_{\kappa_1}^{(+)}|m_{+,\kappa_1}\rangle=\sqrt{m_{+,\kappa_1}}|m_{+,\kappa_1}-1\rangle,$$

$$\alpha_{\kappa_1}^{\dagger(+)}|m_{+,\kappa_1}\rangle=\sqrt{m_{+,\kappa_1}+1}|m_{+,\kappa_1}+1\rangle.\tag{47}$$

The k integration is sometimes replaced by the following discrete summation:

$$dk \leftrightarrow \frac{2\pi}{C}\sum_{\kappa} . \tag{48}$$

When we calculate the transition amplitude, we must square the matrix element. Thus, one $\delta_p(k_1+k_2+k_3+k_4)$ is replaced as follows:

$$\delta_p(k_1+k_2+k_3+k_4) \rightarrow \frac{\Delta}{2\pi}N. \tag{49}$$

One $2\pi\delta(2\pi lN_c/C+\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_3}^{(+)}+\omega_{k_4}^{(+)})$ is replaced by the time interval "t." This "t" is removed in Eq. (8).

When the β functions do not have an s dependence, l is always zero. This means that heating does not occur, because in this case the summation of the phonon frequency must always be zero. In the accelerator that we consider, the δ function is satisfied when $l=-1$. The heating rate that comes from the terms $a_{k_1}^{(+)\dagger}a_{k_2}^{(+)\dagger}a_{k_3}^{(+)\dagger}a_{k_4}^{(+)\dagger}$ and $a_{k_1}^{(+)}a_{k_2}^{(+)}a_{k_3}^{(+)}a_{k_4}^{(+)}$ causes the following resonance condition:

$$\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_3}^{(+)}+\omega_{k_4}^{(+)}=2\pi N_c/C. \tag{50}$$

Though the other parts of the interaction Hamiltonian cause the other resonance condition, we do not have to calculate them for our accelerator, because the phonon frequencies cannot satisfy these conditions, for example, $\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_3}^{(3)}-\omega_{k_4}^{(3)}=2\pi lN_c/C$.

By taking a summation on the number of initial phonons weighting the Boltzmann factor and on that of the final phonons,

$$\begin{aligned}
&\sum_{\kappa_1,\kappa_2,\kappa_3,\kappa_4}\ \sum_{\kappa_1',\kappa_2',\kappa_3',\kappa_4'}\ \sum_{m,n}\big(\langle n_{+,1},\ldots,n_{3,N}|\alpha_{\kappa_1}^{(+)\dagger}\alpha_{\kappa_2}^{(+)\dagger}\alpha_{\kappa_3}^{(+)\dagger}\alpha_{\kappa_4}^{(+)\dagger}|m_{+,1,\ldots,m_{3,N}}\rangle\\
&\times\overline{\langle n_{+,1,\ldots,}n_{3,N}|\alpha_{\kappa_1'}^{(+)\dagger}\alpha_{\kappa_2'}^{(+)\dagger}\alpha_{\kappa_3'}^{(+)\dagger}\alpha_{\kappa_4'}^{(+)\dagger}|m_{+,1},\ldots,m_{3,N}\rangle}-\langle n_{+,1},\ldots,n_{3,N}|\alpha_{\kappa_1}^{(+)}\alpha_{\kappa_2}^{(+)}\alpha_{\kappa_3}^{(+)}\alpha_{\kappa_4}^{(+)}|m_{+,1,\ldots,m_{3,N}}\rangle\\
&\times\overline{\langle n_{+,1},\ldots,n_{3,N}|\alpha_{\kappa_1'}^{(+)}\alpha_{\kappa_2'}^{(+)}\alpha_{\kappa_3'}^{(+)}\alpha_{\kappa_4'}^{(+)}|m_{+,1},\ldots,m_{3,N}\rangle}\big)\frac{v\hbar}{k_BNT}(\omega_{\kappa_1}^{(+)}+\omega_{\kappa_2}^{(+)}+\omega_{\kappa_3}^{(+)}+\omega_{\kappa_4}^{(+)})\\
&\times\left(1-\exp\left[-\frac{v\hbar\omega_1^{(+)}}{k_BT}\right]\right)\cdots\left(1-\exp\left[-\frac{v\hbar\omega_N^{(3)}}{k_BT}\right]\right)\exp\left[-m_{+,1}\frac{v\hbar\omega_1^{(+)}}{k_BT}\cdots-m_{3,N}\frac{v\hbar\omega_N^{(3)}}{k_BT}\right]\\
&=\sum_{\kappa_1,\kappa_2,\kappa_3,\kappa_4}24\frac{\exp\left[\frac{v\hbar(\omega_{\kappa_1}^{(+)}+\omega_{\kappa_2}^{(+)}+\omega_{\kappa_3}^{(+)}+\omega_{\kappa_4}^{(+)}-1)}{k_BT}\right]-1}{\left(\exp\left[\frac{v\hbar\omega_{\kappa_1}^{(+)}}{k_BT}\right]-1\right)\left(\exp\left[\frac{v\hbar\omega_{\kappa_2}^{(+)}}{k_BT}\right]-1\right)\left(\exp\left[\frac{v\hbar\omega_{\kappa_3}^{(+)}}{k_BT}\right]-1\right)\left(\exp\left[\frac{v\hbar\omega_{\kappa}^{(+)}}{k_BT}\right]-1\right)}\\
&\times\frac{v\hbar}{k_BNT}(\omega_{\kappa_1}^{(+)}+\omega_{\kappa_2}^{(+)}+\omega_{\kappa_3}^{(+)}+\omega_{\kappa_4}^{(+)}).
\end{aligned}\tag{51}$$

By taking the limit $\hbar\rightarrow 0$ and performing integration, we obtain the heating rate,

$$\frac{1}{T}\frac{dT}{ds}=\int_{-\pi/\Delta}^{\pi/\Delta}dk_1dk_2f(k_1,k_2), \tag{52}$$

where

$$\begin{aligned}
f(k_1,k_2)=&\frac{r_M^2}{384\pi^2(\gamma^2-1)^2\Delta^7}\left(\frac{k_BT}{Mv^2}\right)^2\frac{(\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_{3n}}^{(+)}+\omega_{-k_1-k_2-k_{3n}}^{(+)})^2}{\omega_{k_1}^{(+)}\omega_{k_2}^{(+)}\omega_{k_{3n}}^{(+)}\omega_{-k_1-k_2-k_{3n}}^{(+)}}\times\frac{|9A_{-1}+72B_{-1}|^2}{\left|\frac{d\omega_{k_{3n}}^{(+)}}{dk_{3n}}-\frac{d\omega_k^{(+)}}{dk}\Big|_{k=-k_1-k_2-k_{3n}}\right|}\\
&\times\Bigg[\sum_{m=1}^{\infty}\frac{1}{m^5}\{1-\cos k_1m\Delta-\cos k_2m\Delta-\cos k_{3n}m\Delta-\cos(k_1+k_2+k_{3n})m\Delta+\cos(k_1+k_2)m\Delta\\
&+\cos(k_2+k_{3n})m\Delta+\cos(k_{3n}+k_1)m\Delta\}\Bigg]^2,
\end{aligned}\tag{53}$$

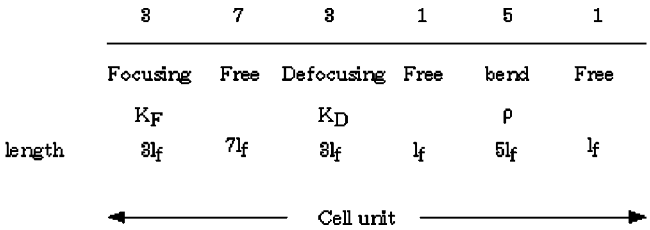

FIG. 2. Lattice configuration of the unit cell. Each parameter is written in Table I.

k_{3n} are all values that satisfy the following conditions:

$$k_{3n}=\mathrm{mod}\left(k_3,\frac{2\pi}{\Delta},-\frac{\pi}{\Delta}\right),$$

$$-\frac{\pi}{\Delta}\leqslant k_1+k_2+k_{3n}<\frac{\pi}{\Delta},$$

$$\omega_{k_1}^{(+)}+\omega_{k_2}^{(+)}+\omega_{k_{3n}}^{(+)}+\omega_{-k_1-k_2-k_{3n}}^{(+)}=\frac{2\pi N_c}{C}. \tag{54}$$

III. COMPARISON WITH OUR THEORY AND SIMULATION RESULTS FOR A ONE-DIMENSIONAL CRYSTAL

According to Eq. (4), we can expect that the heating rate will be reduced by N_c^{-2} when we make N_c larger with fixed tunes and circumference. We also expect that T_c is almost the same for any N_c, because there is no N_c dependence in Eq. (7). Further, we expect that we can reduce the heating rate in the lower temperature region by increasing N_c. According to Eqs. (41), (44), and (53), we can evaluate the N_c dependence of the heating rate. Since $n\omega=2\pi N_c/C$, we suppose that each frequency is proportional to N_c, where n is the index of resonance. When $\theta_0^k\simeq 0$,

$$A_l\propto\frac{1}{N_c^{n/2}}. \tag{55}$$

We thus suppose that

$$\frac{1}{T}\frac{dT}{ds}\propto\frac{1}{N_c^{2n-1}}. \tag{56}$$

We therefore expect that we can reduce the heating rate by increasing the lattice periodicity N_c with fixed tunes.

We calculated the heating rate by a simulation in order to confirm this expectation. We calculated the heating rate as follows. First we reduce the beam temperature by a cooler. If we stop the cooler, the temperature would increase as time passes. Actually, the temperature increases in an oscillating manner, because of the betatron oscillation. Thus, we took an average of the temperature over some lattice periods. After we carried out this average procedure, we calculated the heating rate per lattice period. The parameters of the accelerators are represented in Table I. The configuration of the unit cell is described in Fig. 2. The results are represented in

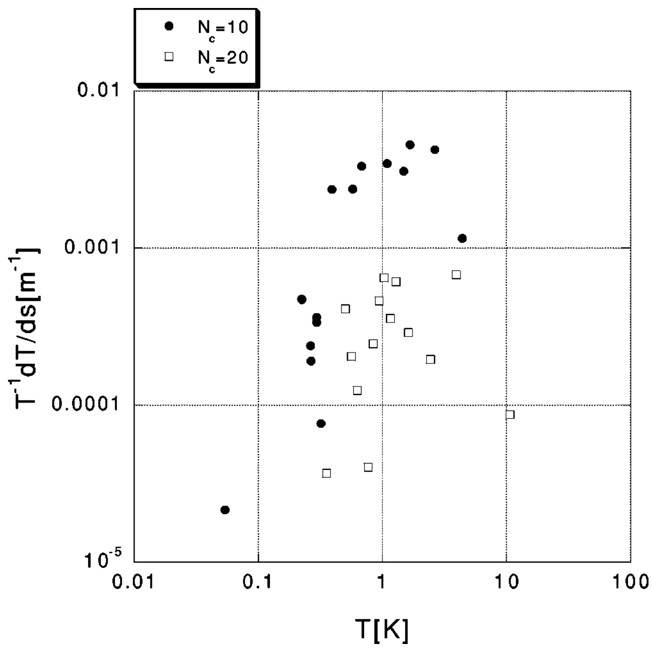

FIG. 3. Heating rate for different N_c.

Fig. 3. According to Eq. (7), $T_c=1.16$ K. We can see the tendency that the heating rate becomes smaller, as N_c becomes larger. For the $N_c=9$ case, we actually calculated the heating rate using our theory. For this number density, the crystalline beam has a chain structure (See Fig. 4). In Fig. 5, we present the dispersion relation of $\omega_k^{(+)}$, $\omega_k^{(-)}$, and $\omega_k^{(3)}$ and Fourier coefficients of beam oscillations obtained by the tracking. The results of theory are consistent with the simulation results (see Fig. 6).

IV. CONCLUSIONS AND DISCUSSIONS

We present here a theory that explains the heating rate of hadron beams. We evaluate the heating rate at higher tem-

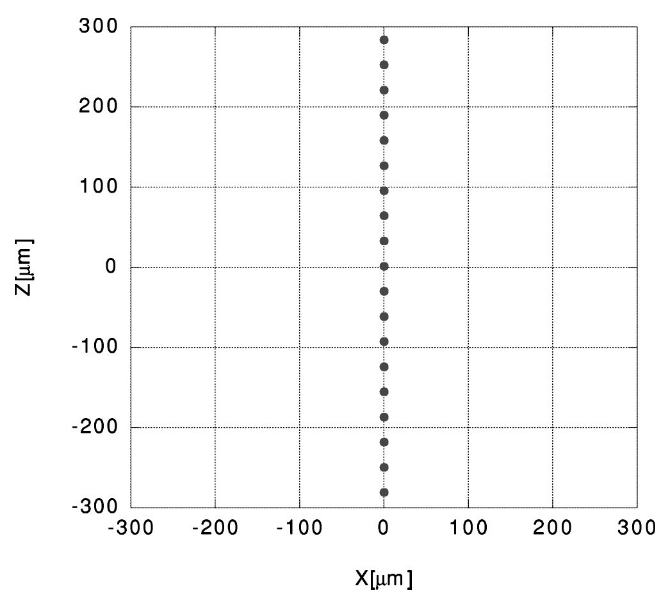

FIG. 4. A crystalline beam when the number density is 31 860.9 m^{-1}

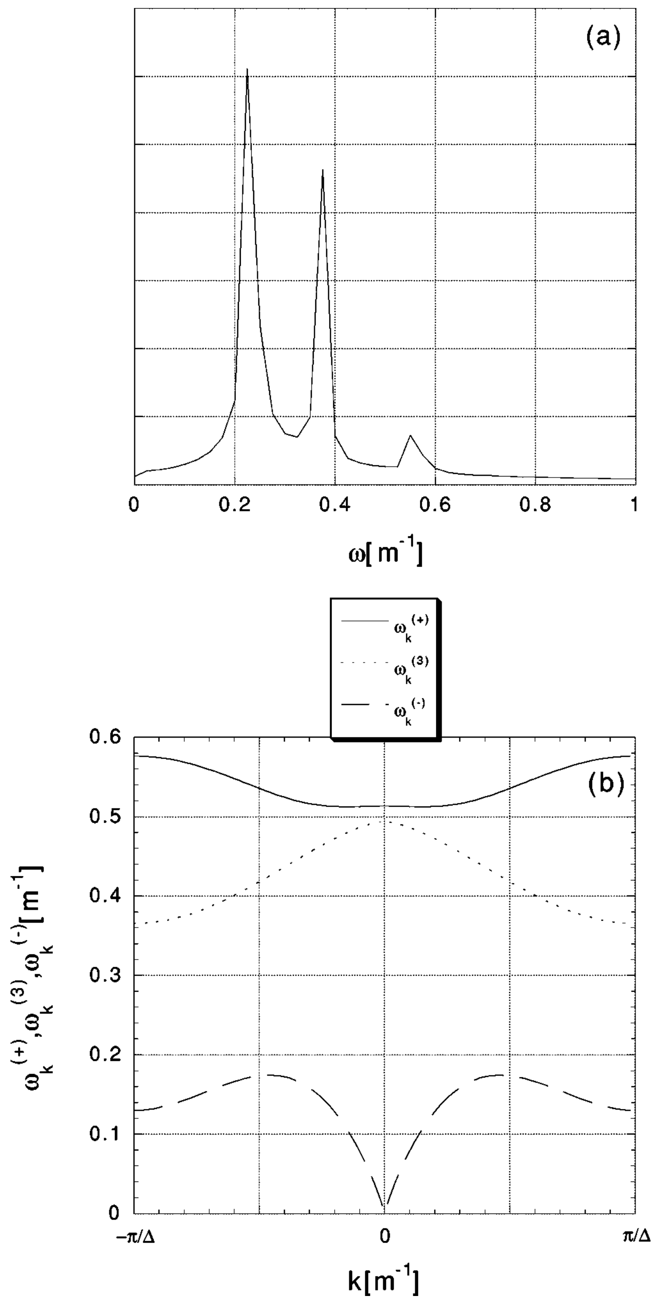

FIG. 5. Spectrum of the beam frequency obtained by tracking (a), and dispersion relation of $\omega_k^{(+)}$, $\omega_k^{(-)}$, and $\omega_k^{(3)}$ obtained by theory (b).

perature using a theory based on intrabeam scattering. We find that we can reduce the heating rate caused by intrabeam scattering by increasing the lattice periodicity.

However, beam heating at a lower temperature is caused by another mechanism, not intrabeam scattering. If the heating rate increases upon increasing the lattice periodicity, we cannot reduce the peak value of the heating rate. We thus, need a theory to explain beam heating at a lower temperature. For a lower temperature, we formulate a theory from a quantum point of view. We can expect that the hadron beam is crystallized at a lower temperature. This crystalline beam vibrates around its equilibrium orbit. This kind of vibration is

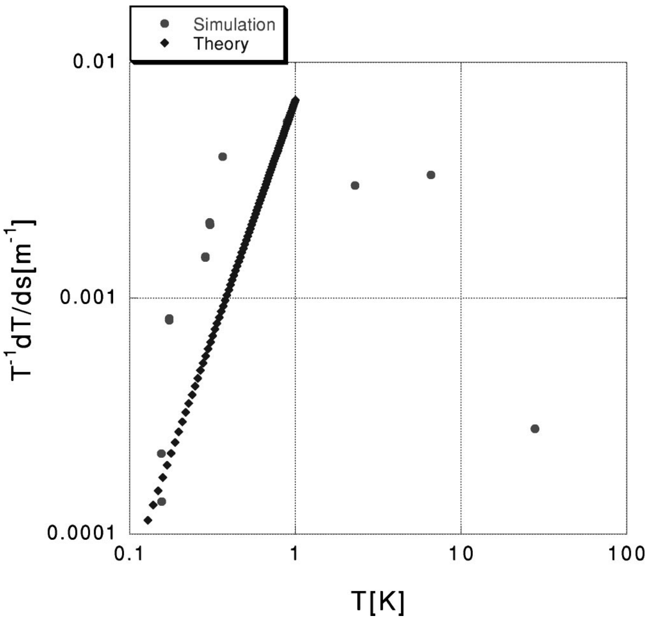

FIG. 6. Heating rate obtained by a simulation and theory.

equivalent to phonon emissions in a quantum picture. If the phonon transits from lower orbits to higher orbits, heating occurs. Actually, the heating at lower temperatures is caused by a resonance between the lattice periodicity and phonon emission of the crystal lattice. According to this theory, we expect that the heating rate for a lower temperature part can be reduced by increasing the lattice periodicity of the accelerator. Our theory explains well the simulation results. In order to make crystalline beams, it is better to construct an accelerator that can reduce the maximum of the heating rate. By increasing the lattice periodicity, the power of the laser that cools hadron beams can be reduced.

Further, our theory predicts a quantum effect in hadron beams. When $\hbar v\omega/k_B > T$, T dependence in the heating rate is distorted from a polynomial of T. By finding this distortion in the heating-rate curve, we will find the quantum effect in beam physics.

In this paper, we explicitly calculated the heating rate for the case that the crystal beam is one-dimension case. In two or three dimension, we expect that the heating rate has the form $T^{-1}\,dT/ds = \alpha + \beta T$. Thus, the heating rate in these dimensions can be larger than that in one dimension. In order to calculate it for these cases, we need more approximations or models. It is necessary to calculate the heating rate for these dimensions, and confirm that we can reduce the heating rate by increasing N_c for those dimensions, too.

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APPENDIX A: DIFFUSION TIME

The diffusion rate of the transverse and longitudinal emittances ϵ_x, ϵ_y, ϵ_s, caused by intrabeam scattering is given as follows [12]:

$$\frac{d}{dt}\ln\epsilon_a=\frac{\pi^2\alpha_e^2MN(\log)}{\gamma\tilde{\Gamma}}\left\langle\int_0^\infty\frac{d\lambda\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}}\times\left\{\mathrm{Tr}\,L^{(a)}\,\mathrm{Tr}\left(\frac{1}{L+\lambda I}\right)-3\,\mathrm{Tr}\,L^{(a)}\left(\frac{1}{L+\lambda I}\right)\right\}\right\rangle, \tag{A1}$$

where a represents x, y, and s and the matrices are given as follows:

$$L^{(x)}=\frac{\beta_x}{\epsilon_x}\begin{pmatrix}1 & -\gamma\phi & 0\\ -\gamma\phi & \frac{\gamma^2\eta^2}{\beta_x^2}+\gamma^2\phi^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{A2}$$

$$L^{(s)}=\frac{\gamma^2}{\epsilon_s}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \tag{A3}$$

$$L^{(y)}=\frac{\beta_y}{\epsilon_y}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \tag{A4}$$

$$\phi=\eta'-\frac{\beta_x'\eta}{2\beta_x}, \tag{A5}$$

$L=L^{(x)}+L^{(y)}+L^{(s)}$, I is the identity matrix and prime denotes the differentiation of s. The brackets $\langle\cdots\rangle$ denote the average around the ring. The diffusion rate of the products of these emittances is given as

$$\frac{d}{dt}\ln\epsilon_s\epsilon_x\epsilon_y\propto\left\langle(\lambda_1-\lambda_2)^2\int_0^\infty\frac{d\lambda\lambda^{1/2}}{(\lambda_1+\lambda)^{3/2}(\lambda_2+\lambda)^{3/2}(\lambda_3+\lambda)^{1/2}}+\text{two cyclic permutations}\right\rangle, \tag{A6}$$

where λ_1, λ_2, and λ_3 are eigenvalues of L.

APPENDIX B: TAYLOR SERIES OF THE COULOMB POTENTIAL FOR ONE-DIMENSION CASE

Here, we explicitly present a Taylor series of the Coulomb potential for the one-dimension case up to fourth order. The potential V_c is written as

$$\begin{aligned}V_c=\frac{1}{2}\sum_{i=-\infty}^{\infty}\sum_{m\neq 0}\Bigg\{&\frac{1}{|z_{i+m}^{(0)}-z_i^{(0)}|}+\frac{1}{2}\Bigg[-\frac{\beta_x(s)}{|z_{i+m}^{(0)}-z_i^{(0)}|^3}\\&\times(X_{i+m}-X_i)^2-\frac{\beta_y(s)}{|z_{i+m}^{(0)}-z_i^{(0)}|^3}(Y_{i+m}-Y_i)^2\\&+\frac{2}{|z_{i+m}^{(0)}-z_i^{(0)}|^3}(\delta z_{i+m}-\delta z_i)^2\Bigg]\\&+\frac{1}{24}\Bigg[\frac{9(\beta_x(s)(X_{i+m}-X_i)^2+\beta_y(s)(Y_{i+m}-Y_i)^2)^2}{|z_{i+m}^{(0)}-z_i^{(0)}|^5}\\&+\frac{24(\delta z_{i+m}-\delta z_{i+m})^4}{|z_{i+m}^{(0)}-z_i^{(0)}|^5}\\&-\frac{72\beta_x(s)(X_{i+m}-X_i)^2(\delta z_{i+m}-\delta z_i)^2}{|z_{i+m}^{(0)}-z_i^{(0)}|^5}\\&-\frac{72\beta_y(s)}{|z_{i+m}^{(0)}-z_i^{(0)}|^5}(Y_{i+m}-Y_i)^2(\delta z_{i+m}-\delta z_i)^2\Bigg]\Bigg\}.\end{aligned} \tag{B1}$$

The odd coefficients of the Taylor series are always zero because of their symmetries.

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